Fundamental groups of klt varieties

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1. Fundamental groups in Algebraic Geometry

- X normal algebraic variety over $\mathbb{C}$

we can view X as top space with the Euclidean metric and define the
<u>topological fundamental group</u>

$$\pi_1(X) := \{\text{closed loops } \gamma : [0,1] \to X\} \Big/ \text{homotopy equiv.}$$

<u>Problem</u>: $\pi_1(X)$ need not be algebraic, nor need the associated <u>universal cover</u>

$$\pi_1(X) \circlearrowright \tilde{X} \xrightarrow[/\pi_1(X)]{} X$$

$\pi_1(X)$

be algebraic.

Solution: consider finite, étale covers

$$f_i : X_i \to X \quad , i \in I$$

I is a directed set by $j \le i :\Leftrightarrow \exists\, \underset{\text{finite, étale}}{f_{ij} : X_i \to X_j}$

f_{ij} yield grouphom $g_{ij} : \underset{\overset{\|}{G_i}}{\mathrm{Aut}_X(X_i)} \longrightarrow \mathrm{Aut}_X(X_j)$

$\rightsquigarrow$ get a proj system of groups $\{(G_i)_{i\in I} ; (g_{ij})_{j \le i}\}$

Define the <u>étale fundamental group</u>

$$\hat{\pi}_1(X) := \varprojlim_{i\in I} G_i = \{ g \in \prod_I G_i \mid g_j = g_{ij}(g_i) \ \forall j \le i \}$$

Easy to see: $\hat{\pi}_1(X)$ is the <u>profinite completion</u> of $\pi_1(X)$ $\left(\hat{G} = \varprojlim G/N \right)$ N finite index

Attention:
- in general, there is no associated "universal étale cover" in the category of algebraic varieties

- there is a natural grouphom
$$\pi_1(X) \to \hat{\pi}_1(X)$$
with dense image, but not nec inj or surj

E.g.:
- G may be very bad but have no normal subgroup of finite index $\Rightarrow \hat{G}$ trivial
- G may be infinite but rather well-behaved and $\hat{G}$ may be nasty:
$$X = \mathbb{C}^* \simeq \mathbb{R} \times S_1 \rightsquigarrow \pi_1(X) = \mathbb{Z}$$
$$\tilde{X} = \mathbb{R}^2$$
but $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$
$$= \prod_p \mathbb{Z}_p$$
$\mathbb{Z}_p$ p-adic integers

- the best situation is of course when $\pi_1(X)$ is finite

2. Fundamental groups of singularities

~ two viewpoints

global: if X is smooth, then a cover is étale iff it is étale in codim 1 maybe we should allow covers to ramify in codim 2 for singular X ?!

equiv: over X_{sing}

$\rightsquigarrow$ consider $\pi_1(X_{reg})$

local: (X,x), X locally at x is contractible, so we should at least allow ramification over x

$\rightsquigarrow$ consider local fundamental group

$$\pi_1^{loc}(X,x) := \pi_1(X \setminus \{x\})$$

maybe better regional fund group

$$\pi_1^{reg}(X,x) = \pi_1(X_{reg})$$

For a log pair (X,Δ), we may allow covers that ramify over Δ "in a controlled manner", i.e.

$$f:(Y,\Delta_Y) \to (X,\Delta)$$

s.th. Δ_Y is given by $f^*(K_X+\Delta) = K_Y + \Delta_Y$ is effective.

How? decompose $\Delta = \Delta' + \Delta''$

$\Delta' \geq 0$, $\Delta'' = \sum (1-\frac{1}{m_i})\Delta_i$ standard coeffs

$m_i \in \mathbb{N}$

$\mathcal{X} = (X, \Delta^u)$ defines an orbifold structure.

and the orbifold fundamental group

$$\pi_1(X, \Delta^u) = \pi_1(X \setminus \Delta^u) / \langle\langle \gamma_i^{m_i} \rangle\rangle$$

loop around a general point of Δ_i

yields what we want.

3. Fundamental groups of klts and Fanos

Thm (classical) Fano manifolds are simply connected.

less classical (Takayama '99): (X, Δ) log Fano $\Rightarrow$ X simply conn.

Thm (B. 2020)

(1) (X, Δ) log Fano, then $\pi_1(X_{reg}, \Delta|_{X_{reg}})$ finite.

(2) (X, Δ, x) klt sing, then $\pi_1^{reg}(X, \Delta, x)$ finite.

Basic Idea: "local to global induction"

(A) $(1)_{\dim X = n-1} \Rightarrow (2)_{\dim X = n}$

(B) $(2)_{\dim X = n} \Rightarrow (1)_{\dim X = n}$

(A) global to local

basic construction: Kollár component / plt blowup going back to Shokurov/Prokhorov in the 90ies

$\sim$ excep divisor over x which is of Fano type.

<u>Lemma</u> (Existence of plt blowups, Xu 14)

Let $p \in (X, \Delta)$ be a klt point. There exists a $\mathbb{Q}$-divisor H on X and $f: Y \to X$ bir, s.th.:

(1) $\exists$ prime divisor E on Y with $\mathrm{center}_X(E) = p$;

(2) $-(K_Y + f_*^{-1}\Delta + E)$ and $-E$ ample over X

(3) $(X, \Delta + H)$ klt on $X \setminus \{p\}$, $\mathrm{mld}(p, X, \Delta + H) = 0$ and E is unique over p with discr -1

Proof:

- choose general ample L on X passing through p with small coeff's, s.th. $(X, \Delta+L)$ is lc at p and klt on $X\setminus\{p\}$

- log res $g: Z \to (X, \Delta+L)$ s.th. $Ex(g)$ supports a relative ample A

- If we take $0<\delta \ll \varepsilon$ s.th. $\varepsilon g^*L + \delta A \sim_{\mathbb{Q}} L'$ is a general ample $\mathbb{Q}$-div, then there exists $0<t$ s.th. in the formula

$$g^*\left(K_X+\Delta+(t+\varepsilon)L\right) \sim_{\mathbb{Q}} K_Z + g_*^{-1}(\Delta+tL) + L' + \sum a_i E_i$$

there is a unique $a_i = a_1 = 1$ and $a_i < 1$ for $i \geq 2$ and $\mathrm{center}_X(E_1) = p$.

- Now we consider the dlt pair

$$\Big(Z, \underbrace{g_*^{-1}(\Delta+tL) + E_1 + L' + \sum_{i\geq 2} E_i}_{=:\,\Delta_Z}\Big)$$

having $K_Z + \Delta_Z \sim_{X,\mathbb{Q}} \sum (1-a_i) E_i$.

- Now run a $(K_Z+\Delta_Z)$-MMP over X with scaling of L', by [BCHM], it terminates with a good minimal model $h: W \to X$

- $\phi: Z \dashrightarrow W$ contracts precisely the E_i $(i\geq 2)$

so $E_W = \phi_*(E_1)$ is the div part of $\mathrm{Ex}(h)$

- on W $\underbrace{K_W + h_*^{-1}(\Delta + tL) + E_W}_{=:\Delta_W} + \phi_* L' \sim_{X,\mathbb{Q}} 0$

- (W, Δ_W) is plt and for ε sufficiently small, $K_W + \Delta_W + \varepsilon \phi_* L'$ is nef over X.

- Let $f: Y \to X$ be the log canonical model of $(W, \Delta_W + \varepsilon \phi_* L')$;

- $\phi_* A = \lambda E_W$ for some $\lambda > 0$, so we have that

$$-\varepsilon \delta \lambda E_W = \varepsilon \delta \phi_* A \sim_{X,\mathbb{Q}} K_W + \Delta_W + \varepsilon \phi_* L'$$

is nef over X.

- Now define $H := tL + (h \circ \phi)_* L'$

- $W \to Y$ is small and $(Y, E_Y + f_*^{-1}(H + \Delta))$ plt since (W, Δ_W) is.

$-E_Y$ is f-ample and

$$-(K_Y + f_*^{-1}\Delta + E) \sim_{X,\mathbb{Q}} \underbrace{-(1 + a(E, X, \Delta))}_{>0} E$$

is ample as well. $\square$

Now writing

$$\left(K_Y + f_*^{-1}\Delta + E\right)\Big|_E = K_E + \underbrace{\mathrm{Diff}_E f_*^{-1}\Delta}_{=:\Gamma}$$

then the pair (E,Γ) is log Fano

<u>Proof of (A): global to local</u> (for $\frac{1}{n_1}$)

$p \in (X,\Delta)$ klt sing, $f: Y \to X$ plt blowup as above

$$\cdots \to (X_2,p_2) \xrightarrow{\phi_1} (X_1,p_1) \xrightarrow{\phi_0} (X,p)$$

seq of finite morphisms étale over $X \setminus \{p\}$

want to show: $\exists i \in \mathbb{N}$, s.th. ϕ_j trivial $\forall j \geq i$

- Let Y_i be the normalized main component of $X_i \times_X Y$; we have a commutative diagram:

$$\begin{array}{ccc} E_{i+1} \subset Y_{i+1} & \xrightarrow{\psi_i} & Y_i \supset E_i \\ \downarrow f_{i+1} & & \downarrow f_i \\ p_{i+1} \in X_{i+1} & \xrightarrow{\phi_i} & X_i \ni p_i \end{array}$$

- Δ_i = pullback of Δ and Γ_i defined by

$$(K_{Y_i} + \gamma_{i*}\Delta_i + E_i)|_{E_i} = K_{E_i} + \Gamma_i$$

thus $\psi_i|_{E_{i+1}} : (E_{i+1}, \Gamma_{i+1}) \to (E_i, \Gamma_i)$

is log étale in codim 1.

by <u>Assumption</u>: $\exists\, i \in \mathbb{N}$, s.th. $\psi_j|_{E_j}$ is trivial for $j \geq i$, so ψ_j is totally ramified over E_j.

- let γ be a loop around a general point of E_j, cutting Y_j to a surface through this general point, by Mumfords '61 work on surface singularities, we get that γ is torsion in

$$\pi_1(Y_j \setminus E_j) = \pi_1^{loc}(X_j, p_j)$$

so for $k \geq j$ $\phi_j : X_j \to X_{j-1}$ is trivial

$\Box$

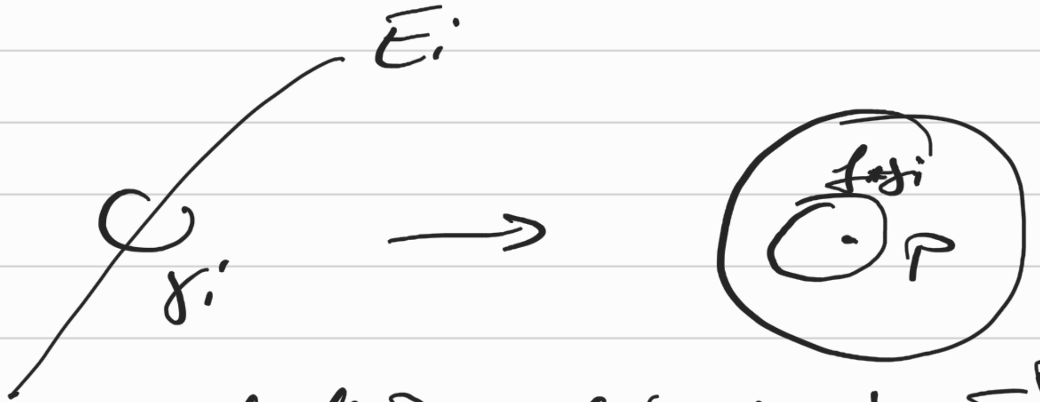
(B) local to global

<u>basic idea:</u> express $\pi_1(Y_{reg}, D)$ as an orbifold fundamental group supported on log res $f: X \to Y$ <u>using</u> the assumption that

π_1^{reg} is finite for klts

π_1 is finite for klts.

<u>How?</u>: f is continuous!
let γ_i a loop around a general point of $E_i \in Ex(f)$



so γ_i is of finite order m_i in $\pi_1^{reg}(X, p)$

thus $$\pi_1(Y_{reg}, D) = \frac{\pi_1\left(X \setminus Ex(f), f_*^{-1}D''\right)}{\langle\langle \gamma_i^{m_i} \rangle\rangle}$$

$$= \pi_1\left(X, f_*^{-1}D'' + \sum\left(1 - \tfrac{1}{m_i}\right)E_i\right)$$

so the following prop will prove part (B):

<u>Proposition</u> Let (Y, D) be log Fano with log res $f: X \to Y$. Then for any choice of $m_i \in \mathbb{N}_{\geq 1}$,

$$\pi_1\left(X, f_*^{-1}D'' + \sum\left(1 - \tfrac{1}{m_i}\right)E_i\right)$$

$=: \Delta$

is finite.

<u>Proof</u>: notation as above, $a_i := \mathrm{disc}(E_i, Y, D)$

$c_i := a_i - \frac{1}{m_i}$

$$L := -f^*(K_Y + D) - \sum_{-1 < c_i < 0} c_i E_i + \sum_{0 \le c_i} (\lceil c_i \rceil - c_i) E_i + f_*^{-1} D''$$

Consider the orbifold $\mathcal{X} = (X, \Delta)$

$K_{\mathcal{X}} = K_X + \Delta$ is the orbifold canonical divisor

$K_{\mathcal{X}} + L = \sum_{0 \le c_i} \lceil c_i \rceil E_i$ is an effective exep divisor on X

<u>Aim</u>: produce many sections on this divisor.

<u>How?</u>: method of Takayama '99 generalized to orbifolds

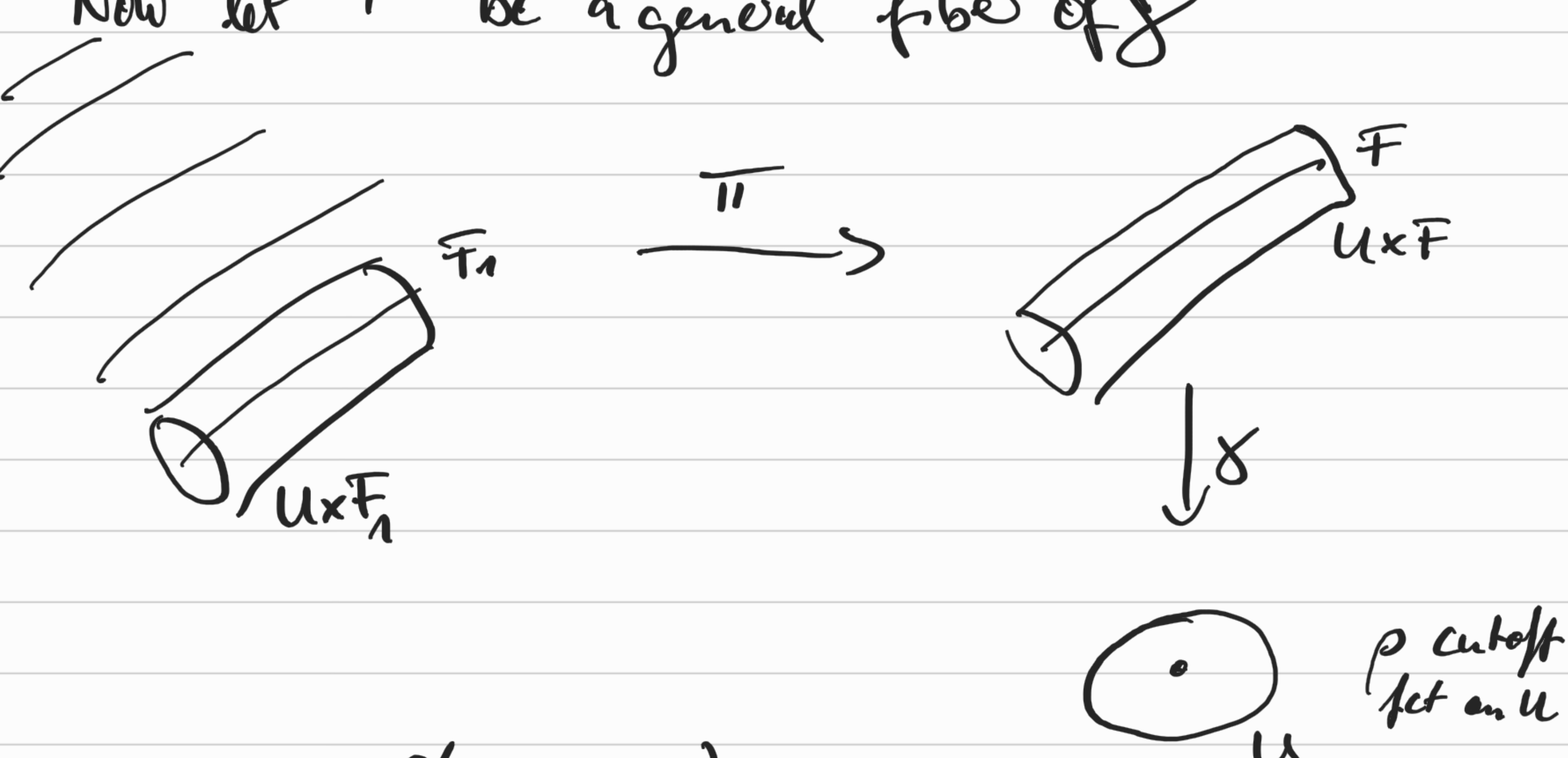
By Claudon '09, the <u>Γ-reduction</u> or <u>Shafarevich map</u> is available for orbifolds, i.e. we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{\tilde{\gamma}} & \Gamma(\tilde{\mathcal{X}}) \\ \big\downarrow {\scriptstyle /\pi_1(\mathcal{X})} & & \big\downarrow {\scriptstyle /\pi_1(\mathcal{X})} \end{array}$$

$$\mathcal{X} \xrightarrow{\gamma} P(\mathcal{X})$$

with:
- $\gamma, \tilde{\gamma}$ are top locally trivial fibrations on open subsets of $X, \tilde{X}$
- γ parametrizes maximal suborbifolds $V \subseteq \mathcal{X}$ with $\mathrm{im}(\pi_1(V) \to \pi_1(\mathcal{X}))$ finite
- $\tilde{\gamma}$ parametrizes maximal compact suborbifolds of $\tilde{\mathcal{X}}$

Now let F be a general fiber of γ



- take $\sigma \in H^0(X, K_{\mathcal{X}} + L)$
- pullback $\tilde{\sigma} := (\pi|_{F_1 \times U})^* (\sigma|_{F \times U}) \cdot (\gamma \circ \pi)^* \rho$
- $\tilde{\sigma}$ is a smooth $\tilde{L} = \pi^* L$ -valued $(n,0)$-form on $\tilde{\mathcal{X}}$

• Now we want to use Nadel vanishing for orbifolds to produce $\nu \in H^0_{(2)}(\tilde{X}, K_{\tilde{X}} + \tilde{L})$

Idea: $\tau = \bar{\partial}(\tilde{\sigma}) = \pi^*\sigma \cdot \bar{\partial}((\gamma \circ \pi^*)\rho)$
is square integrable and $\bar{\partial}$-closed.

$\overset{\text{Nadel}}{\Longrightarrow} \exists\ \omega$ with $\bar{\partial}(\omega) = \tau$, square integrable,
with $\nu = \tilde{\sigma} - \omega$ is nontrivial, square int
and holomorphic, because $\bar{\partial}(\nu) = \tau - \tau = 0$

At this point, we can use the standard theory of Gromov, which gives that

$$P(\nu^{\otimes 2k}) = \sum_{g \in \pi_1(X)} g^* \nu^{\otimes 2k}$$

(the Poincaré series)

for $\pi_1(X)$ infinite, $\exists\ k \in \mathbb{N}$ sts
for two different partitions $k = \sum k_i = \sum k_j'$
$\bigotimes_i P(\nu^{\otimes 2k_i})$ and $\bigotimes_j P(\nu^{\otimes 2k_j'})$

are linearly independent sections of

$$H^0\left(X, (K_X \otimes L)^{\otimes 2k}\right)$$

$\rightsquigarrow$ effective excep

$\Leftarrow$ $\square$

THANKS!